# Storgruppsövning 28/11-13

## 6.4.4

Show that nonirreducible chain may have non-unique stationary distribution.

**Solution:**

$\pi P = \pi$ $\qquad P = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$


Makes any $\pi$ work!

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## 6.4.7

Show that a random walk on a binary tree is transient.

**Solution:**

$X_n$ = level of random walk at time $n$.


$$X_{n+1} = \begin{cases} X_n + 1 & \text{wp } 2/3 \\ X_n - 1 & \text{wp } 1/3 \end{cases} \qquad p_{ij} = \begin{cases} 2/3 & j = i+1 \\ 1/3 & j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

$X_n$ is an ordinary random walk with up probability $2/3$ and down probability $1/3$

ex. 6.2.12, transient by our earlier investigations of random walks.

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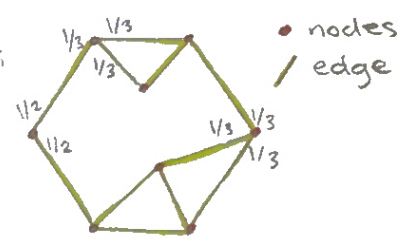
## 6.4.6

Random walk on a finite graph. Particle performs random walk on the nodes (vertices) of a connected random graph. The graph has $\eta$ edges.

We call $d(v)$ is the degree of vertec $v$, meaning that $d(v)$ is the number of edges that connect to vertec $v$. Then we must have $\sum_v d(v) = 2\eta$. Show that $\pi$ given by $\pi_v = d(v)/2\eta$ is a stationary distr.



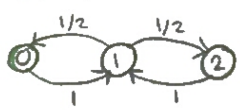

**Solution:**

Check that $\pi$ is PMF – that holds because $\sum_v d(v) = 2\eta \Longrightarrow \sum_v \pi_v = 1$

Can check $\pi P = \pi$. Can also check if $p_{ij}(n) \to \pi_j$ as $n \to \infty$

It is pretty clear that $\pi_j$ must be proportional to $d(j)$. Then we get the suggested formula for $\pi$.

example: look at




$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\pi = \left( \frac{d(0)}{2\eta} \; \frac{d(1)}{2\eta} \; \frac{d(2)}{2\eta} \right) = \left( \frac{1}{4} \; \frac{2}{4} \; \frac{1}{4} \right) \qquad \pi P = \pi$$

6.4.8

At each time $n$ a Poisson with parameter $\lambda$ distributed number of particles $Y_n$ enter a chamber. Lifetimes of particles in champer are geometrically distr. with parameter $p$.
Show that number of particles $X_n$ in chamber is Markov and find $\pi$.

Solution:

$$X_{n+1} = \underbrace{Y_{n+1}}_{\text{new arrivals}} + \underbrace{\sum_{i=1}^{X_n} B_{n,i}}_{\text{how many of them I had survived?}} \quad \text{where } B_{n,i} = \begin{cases} 1 & \text{wp } 1-p \\ 0 & \text{wp } p \end{cases} \Rightarrow \text{Markov}$$

Probability generating function $G_Y(s) = E(s^Y)$

$$\Psi_Y(t) = E(e^{itY}) \qquad \Psi_Y(t) = G_Y(e^{it})$$

$$G_{X_{n+1}}(s) = E(s^{X_{n+1}}) = \sum_{k=0}^{\infty} E(s^{X_{n+1}} \mid X_n = k) P(X_n = k) = \sum_{k=0}^{\infty} E(s^{Y_{n+1} + B_{n,1} + \dots + B_{n,k}}) P(X_n = k)$$

$$= \sum_{k=0}^{\infty} \underbrace{E(s^Y)}_{e^{\lambda(s-1)}} \underbrace{(E(s^B))}_{p+(1-p)s}{}^k P(X_n = k) = e^{\lambda(s-1)} \underbrace{\sum_{k=0}^{\infty} (p+(1-p)s)^k P(X_n = k)}_{G_{X_n}(p+(1-p)s)}$$

At stationarity $G_{X_{n+1}} = G_{X_n} = G_X$

$\Rightarrow G_X(s) = e^{\lambda(s-1)} G_X(p+(1-p)s)$ , solve this for $G_X$ to find stat. dist.

$$g_X(s) = \log G_X(s) \quad , \quad g_X(s) = \lambda(s-1) + g_X(p+(1-p)s)$$

$$g_X(0) = -\lambda + g_X(p)$$

$$g'_X(1) = \lambda + (1-p) g'_X(1) \Rightarrow g'_X(1) = \lambda / p$$

$$g^n_X(1) = (1-p)^n g^n_X(1) \Rightarrow g^n_X(1) = 0 \, , \; n \geq 2$$

$$\Rightarrow g_X(s) = \frac{\lambda}{p}(s-1) \Rightarrow G_X(s) = e^{\frac{\lambda}{p}(s-1)} \Rightarrow X \text{ is } Po\left(\tfrac{\lambda}{p}\right)$$

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6.5.1

Random walk on the set $\{0, 1, \dots, b\}$ has transition matrix given by $P_{00} = 1 - \lambda_0$ , $P_{bb} = 1 - \mu_b$ , $P_{i,i+1} = \lambda_i$ and $P_{i+1,i} = \mu_{i+1}$ where $\lambda_i + \mu_i = 1$.
Show that chain is time reversible (means that $Y_n = X_{N-n}$ has same $P$ as $X$ itself.).

Solution:

Time reversible if $\pi_i P_{ij} = \pi_j P_{ji}$ , here this means $\pi_j \lambda_j = \pi_{j+1} \mu_{j+1}$

($i = j$, $j = j+1$ ; $j = j+1$, $i = j$)

$$\pi_{j+1} = \pi_0 \frac{\lambda_0}{\mu_1} + \dots + \frac{\lambda_j}{\mu_{j+1}} \quad \text{when } \pi_0 \text{ is chosen to make } \pi \text{ a PMF.}$$

6.5.6 a)

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$ time reversible?

Solution:
Check if $\underset{0}{\pi_i}\, \underset{0\,1}{P_{ij}} = \underset{1}{\pi_j}\, \underset{1\,0}{P_{ji}}$ $\quad \pi_0 \alpha = \pi_1 \beta \Longrightarrow \pi = \left(\frac{\beta}{\alpha+\beta} \;\; \frac{\alpha}{\alpha+\beta}\right)$
is stat. dist. and chain is time reversible.

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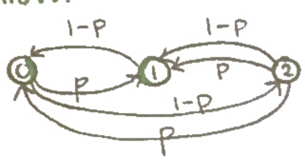
6.5.6 b)

$$P = \begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$$ time reversible?

Solution:




$\Longrightarrow p = \left(\frac{1}{3}\;\frac{1}{3}\;\frac{1}{3}\right)$

$\frac{1}{3} \underset{\substack{0\,1\\0\,2\\1\,2}}{P_{ij}} = \frac{1}{3} \underset{\substack{1\,0\\2\,0\\2\,1}}{P_{ji}} \Longrightarrow p = 1-p \Longrightarrow p = 1/2$ time reversible

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6.5.2
Show $\underline{X}$ is time reversible iff
$\underset{j}{P_{j_1 j_2}} P_{j_2 j_3} \cdots P_{j_{n-1} j_n} \underset{j\;i}{P_{j_n j_1}} = \underset{i\;j}{P_{j_1 j_n}} \underset{j}{P_{j_n j_{n-1}}} \cdots P_{j_3 j_2} \underset{i}{P_{j_2 j_1}}$ all choices of $j_1, \dots, j_n$.
Solution:
By summation of all possible values of $j_2, \dots, j_{n-1}$ we get

$\underset{i\;j}{P_{j_1 j_n}}(n-1)\, \underset{j\;i}{P_{j_n j_1}} = \underset{i\;j}{P_{j_1 j_n}}\, \underset{j\;i}{P_{j_n j_1}}(n-1)$ $\quad \boxed{\Uparrow}$

$\big\downarrow n\to\infty \qquad \big\downarrow n\to\infty$

$\underset{j}{\pi_{j_n}}\, \underset{j\;i}{P_{j_n j_1}} \qquad \underset{i\;j}{P_{j_1 j_n}}\, \underset{i}{\pi_{j_1}}$ $\quad \boxed{\Downarrow}$ similar...